Math 564: Advance Analysis 1
Lecture 24

Characterization of completeness. A normed vector space X is complete if and only if every absolutely convergent series converges (in norm).

Proof. $\Rightarrow$. Suppose $(x_n) \subseteq X$ has absolutely convergent series, i.e. $\sum_n \|x_n\| < \infty$. We need to show the sequence $\sum_{n<N} x_n$ of partial sums is Cauchy.

$$\Big\| \sum_{n<N} x_n - \sum_{n<N+M} x_n \Big\| = \Big\| \sum_{n=N}^{N+M} x_n \Big\| \leq \sum_{n=N}^{N+M} \|x_n\| \leq \sum_{n=N}^{\infty} \|x_n\| \to 0 \text{ as } N \to \infty.$$

$\Leftarrow$. Suppose every abs. convergent series converges and let $(x_n) \subseteq X$ be a Cauchy sequence. By the Cauchy property, it's enough to prove that a subsequence of (x_n) converges in X. WLOG, by switching to a subsequence, we may assume that $\forall n$, $\|x_n - x_{n+1}\| < 2^{-n}$. Then let $y_n := x_n - x_{n-1}$, where $y_0 := 0 \in X$. Observe that $\sum_{i=0}^{n} y_i = 0 + (\cancel{x_0} - 0) + (x_1 - \cancel{x_0}) + \dots + (x_n - \cancel{x_{n-1}}) = x_n$ and also $\sum_{i=0}^{\infty} \|y_i\| = \sum_{i=0}^{\infty} \|x_i - x_{i-1}\| = \|x_0\| + \sum_{i\geq 1} \|x_i - x_{i-1}\| \leq \|x_0\| + \sum_{i \geq 1} 2^{i} < \infty$.

Thus, $\sum_{i=0}^{\infty} y_i$ converges and that's the limit of (x_n). $\square$

Def. A normed vector space is called a **Banach space** if it is complete.

Examples. (a) $\mathbb{R}^d$, $\mathbb{C}^d$

(b) $M_{n \times m}(\mathbb{R})$ with the operator norm.

(c) $C([0,1])$ with the uniform norm.

Cor. $L^1(X,\mu)$ is a Banach space for any measure space (X,μ).

Proof. Suppose $\sum_{n\in\mathbb{N}} \|f_n\|_1 < \infty$ and show that $\sum_{n\in\mathbb{N}} f_n$ converges in L^1 norm. By the MCT, we have that $\int \sum_n |f_n| = \sum_n \int |f_n| = \sum_n \|f_n\|_1 < \infty$.

So the function $g := \sum_n |f_n| \in L^1$, in particular, $g < \infty$ a.e.
Then for a.e. $x \in X$, the series $\sum_n f_n(x)$ converges absolutely:
$$\sum_n |f_n(x)| = g(x) < \infty.$$

Hence it converges to some real, call it $f(x)$. In other words, the partial sums $\sum_{n<N} f_n \to f$ pointwise as $N \to \infty$.

Note that $\left|\sum_{n<N} f_n\right| \leq \sum_{n<N} |f_n| \leq g \in L^1$, so by the DCT, $\sum_{n<N} f_n \xrightarrow{L^1} f$. □

We'll give other examples shortly.

Linear transformations. Given two normed vector spaces X, Y, the natural morphisms between X, Y are continuous linear transformations $T : X \to Y$.

Slogan (Tao). Weak regularity + group structure $\Rightarrow$ strong regularity.

Def. A linear transformation $T : X \to Y$ between normed vector spaces is called **bounded** if it is Lipschitz, i.e. $\exists\, C > 0$ s.t. $\forall x \in X$
$$\|Tx\| \leq C \cdot \|x\|.$$

Prop. For a linear transformation T, the following are equivalent:
(1) T is continuous at $0 \in X$.
(2) T is continuous.
(3) T is bounded.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Trivial.

Extra: (1) $\Rightarrow$ (2). Let $x \in X$ and let $x_n \to x$. Then $x - x_n \to 0$, so $T(x - x_n) \to T(0) = 0$, but $T(x - x_n) = Tx - Tx_n$ so $\|Tx_n - Tx\| \to 0$, hence $Tx_n \to Tx$ in the Y-norm.

(1)$\Rightarrow$(3). Suppose towards a contradiction, that $\forall n \in \mathbb{N}^+ \; \exists x_n$ s.t.
$$\|Tx_n\| > n \cdot \|x_n\|.$$
Divide both sides by $\|x_n\|$, we get $\left\|T\left(\frac{x_n}{\|x_n\|}\right)\right\| > n$, so we may assume that $\|x_n\| = 1$ and we have
$$\|Tx_n\| > n, \text{ equivalently, } \left\|T\left(\frac{x_n}{n}\right)\right\| > 1. \; (*)$$
But $\left\|\frac{x_n}{n}\right\| = \frac{1}{n} \to 0$ so by the continuity of T at $0 \in X$

we have $T\left(\frac{x_n}{n}\right) \to 0$, contradicting $(*)$. $\square$

For any linear $T : X \to Y$, let $\|T\| := \sup\limits_{\|x\|=1} \|Tx\| = \sup\limits_{x \neq 0} \frac{\|Tx\|}{\|x\|}$, call this the norm of T. This is indeed a norm:

(i) $\|T\| = 0 \iff Tx = 0 \; \forall x \in X \iff T = 0$,

(ii) $\|cT\| = |c| \cdot \|T\|$ for all $c \in \mathbb{R}$.

(iii) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$ because $\|(T_1 + T_2)(x)\| = \|T_1 x + T_2 x\| \leq \|T_1 x\| + \|T_2 x\|$
$\leq \|T_1\| \cdot \|x\| + \|T_2\| \|x\| = (\|T_1\| + \|T_2\|) \|x\|$.

The norm also satisfies the following: If $T : X \to Y$ and $S : Y \to Z$, then

(iv) $\|S \circ T\| \leq \|S\| \cdot \|T\|$. Indeed, $\|S \circ T(x)\| = \|S(Tx)\| \leq \|S\| \cdot \|Tx\| \leq \|S\| \cdot \|T\| \cdot \|x\|$.

Letting $L(X, Y)$ denote the space of all bdd linear transformations, equipped with this norm, becomes a normed vector space itself.

Prop. If Y is Banach space, then $L(X, Y)$ is a Banach space.

Proof. HW

The space $L(X, X)$ is also denoted by $B(X)$ and (iv) makes this a normed algebra, heavily studied in functional analysis, in particular C^*-algebras and von Neumann algebras.

An important space is $L(X, \mathbb{R})$, whose elements are called bdd linear functionals. This space is called the dual space of X and denoted by, e.g., X^* or $\hat{X}$ or something else.

L^p spaces. For $d \in \mathbb{N}^+$, consider the set $d := \{0, 1, \ldots, d-1\}$ as a measure space with the counting measure. Because all subsets of d are measurable, $L(d)$ = the set of all functions $d \to \mathbb{R}$, i.e. $L^1(d) = \mathbb{R}^d$, equipped with the norm $\|f\|_1 := \sum_{i \in d} |f_i|$, where $f := (f_0, f_1, \ldots, f_{d-1})$. But $\mathbb{R}^d$ is studied using other norms $\|\cdot\|_p$ where $1 \le p \le \infty$ and $\|f\|_\infty := \max_{i<d} |f_i|$. These norms are all equivalent to each other (i.e. bi-Lipschitz) because d is finite. However, even taking $d := \mathbb{N}$ creates different spaces with these different norms.

Def. Let (X, μ) be a measure space, $p > 0$. Let $L^p(X, \mu)$ denote the set of measurable functions $X \to \mathbb{R}$ such that $|f|^p \in L^1(X, \mu)$. Again, we consider two functions equal if they agree a.e.

Prop. $L^p(X, \mu)$ is a vector space.

Proof. We only check closure under +. Let $f, g \in L^p := L^p(X, \mu)$.

$$|f+g|^p \le (2\max(|f|,|g|))^p = 2^p \cdot \max(|f|,|g|)^p \le 2^p (|f|^p + |g|^p),$$ so

$$\| |f+g|^p \|_1 \le 2^p \| |f|^p \|_1 + 2^p \| |g|^p \|_1 < \infty. \qquad \square$$

Define $\|\cdot\|_p : L^p(X, \mu) \to [0, \infty)$ by $\|f\|_p := \left(\int |f|^p\right)^{\frac{1}{p}}$.

Prop. For $p \in (0,1)$, $\|\cdot\|_p$ is not a norm.

Proof. Note that for any $a, b > 0$, we have $a^p + b^p > (a+b)^p$.

(Ad hoc proof. For $t > 0$, $t^{p-1} > (a+t)^{p-1}$, $\int_0^b t^{p-1}\,dt > \int_0^b (a+t)^{p-1}\,dt$,
$\cancel{\tfrac{1}{p}}\, b^p > \cancel{\tfrac{1}{p}}\left((a+b)^p - a^p\right)$, $a^p + b^p > (a+b)^p$.)

Now take disjoint measurable sets $A, B \subseteq X$ of positive measure, and let $a := \mu(A)^{\frac{1}{p}}$ and $b := \mu(B)^{\frac{1}{p}}$. Thus
$\|\mathbb{1}_A + \mathbb{1}_B\|_p = \left(\int \mathbb{1}_A\,d\mu + \int \mathbb{1}_B\,d\mu\right)^{\frac{1}{p}} = \left(\mu(A) + \mu(B)\right)^{\frac{1}{p}} = \left(a^p + b^p\right)^{\frac{1}{p}} > a + b = \|\mathbb{1}_A\|_p + \|\mathbb{1}_B\|_p$. $\square$

However, we'll see that when $p \geq 1$, $\|\cdot\|_p$ does satisfy the triangle inequality.